Complexity of Equational Theory of Relational Algebras with Standard Projection Elements*

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Abstract

The class TPA of *true pairing algebras* is defined to be the class of relation algebras expanded with *concrete* set theoretical projection functions. The main results of the present paper is that neither the equational theory of TPA nor the first order theory of TPA are decidable. Moreover, we show that the set of all equations valid in TPA is exactly on the Π^1_1 level. We consider the class TPA^- of the relation algebra reducts of TPA's, as well. We prove that the equational theory of TPA^- is much simpler, namely, it is recursively enumerable. We also give motivation for our results and some connections to related work.

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Contents

1 Introduction

The topic of this paper is relevant to the following parts of science: *logic, algebraic logic, recursion theory, and theoretical computer science*. The relevance for the latter is explained e.g. in [12], [41], [43].

The results of this paper concern **t***rue* **p***airing* **a***lgebras* (TPA's for short). The class TPA was introduced, e.g., in Maddux [21, Definitions 17–20], and in different form, independently, in Veloso–Haeberer [42], [43]. TPA's in a different form are also discussed in [40, item 4.1 (iv) on p.96, pp. 251–254]. The purpose for which TPA's are used in these publications is, roughly speaking, to provide a nicer formulation of first order logic with equality. By "nice" here we mean a certain point of view. This point of view was described in various works on logic and algebraic logic, and is often referred to as the *finitization problem*. Section 2 serves for outlining the latter — to the extent we believed it is useful for motivating our results presented in the rest of the sections.

A positive solution for the finitization problem for first order logic (FOL) *without equality* was given in [31], and published later in [32], [33], [35] (among which, the last is the most detailed account on the subject). R. J. Thompson conjectured that there is no positive solution for the finitization problem for FOL with equality. His conjecture is motivated by the fact that he proved that the approach in [35] cannot be extended to FOL with equality, cf. [31, §5], [35, §6]. As it will be explained in subsection 2.4, TPA's were suggested as an alternative approach to solve the finitization problem for FOL with equality.

We will prove that the equational theory Eq(TPA) of TPA's is (not only not recursively enumerable but) such that its degree of unsolvability is exactly Π^1_1. On the other hand, let TPA^- be obtained from TPA by forgetting the constant symbols naming the projection functions. I.e., TPA^- is that reduct of TPA which belongs to the similarity class of relation algebras (RA's). Then Eq(TPA^-) is recursively enumerable but is still not finitely axiomatizable.

In [42, sec.3] the standard models $\mathbf{U}$ of the Extended Calculus of Binary Relations (ECBR for short) are defined. These are relation algebras with some extra operators just as TPA's are such. We will not need to recall the precise definition of these algebras. Let SECBR denote the class of these standard models of ECBR'. Our Corollary 3.3 (after Theorem 3.2) shows that the programme initiated in [42] has certain limitations. It also seems to point in the direction that this approach might not be very convenient for attacking the finitization problem.

Our paper consists of sections 2–7. In section 2 we give a survey of our motivation for our results, exactly stated and proved in the later sections. These results were obtained in 1990. Until now they existed in the form of the research report [22] and were abstracted in [23]. In section 7 we mention some more results concerning the topic of our paper.

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2 Finitization Problem and True Pairing Algebras

2.1 Algebraizing Logic

In algebraic logic, to several kinds of logical systems (*logics* for short) — like classical propositional logic, modal logics, first order logic with or without equality — we associate classes of algebras via an algorithm which is described, e.g., in [15, §4.3, pp. 258–260] or in [4, Part II], [3, section 3], [5], [34, sec.3]. We call such a class of algebras associated to a logic the *algebraic counterpart* of it. For example, the algebraic counterpart of classical propositional logic is the class BA of Boolean algebras. The algebraic counterpart of FOL with equality (using ω many variables) is the class RCA_ω of **r***epresentable* **c***ylindric* **a***lgebras* (of dimension ω), cf. e.g. [15, §4.3] or [4, p. 237].

The point in finding the algebraic counterparts of logics is that logical properties of a logic can often be correlated with algebraic properties of the algebraic counterpart of the logic. Thus not only logical systems can have their algebraic counterparts, but logical properties can have their algebraic counterparts as well.

Going even further, an abstract notion of a logic has been worked out, cf. e.g.

in [7], [4], [5], [3], [34]. Such an abstract logic is a common generalization of the systems mentioned above (and more). We can define (abstract) logical properties in this setting, for example, *interpolation properties*, *definability properties*, *existence of derivation systems*, *decidability* can be formulated in this general setting. Then the algebraic counterparts of these properties can also be formulated. Then one can find algebraic properties which correspond to these logical properties, in the following sense.

$$\begin{gathered}\text{An abstract logic } \mathcal{L} \text{ has logical property } \Phi \\ \Longleftrightarrow \\ \text{its algebraic counterpart } \mathrm{Alg}(\mathcal{L}) \text{ has algebraic property } \mathrm{Alg}(\Phi).\end{gathered}$$

Classical theorems state the equivalence of *Beth definability* of a logic $\mathcal{L}$ and *surjectiveness of epis* in the category formed by $\mathrm{Alg}(\mathcal{L})$ and the homomorphisms between them (cf. e.g. [4, Thm.58]). Another example for such results is that the Craig interpolation property of $\mathcal{L}$ is equivalent with the amalgamation property of $\mathrm{Alg}(\mathcal{L})$ (a precise formulation is, e.g., [4, Thm.62]).

2.2 Hilbert style axiomatizability versus finitely based variety

Theorems in the spirit of the ones mentioned in the previous subsection connect properties of derivation systems of a logic with axiomatizability properties of the corresponding classes of algebras. Such a theorem, interesting from the point of view of the present paper, states that

$$\begin{gathered}\text{A logic } \mathcal{L} \text{ has a truly finite Hilbert style inference system} \\ \Longleftrightarrow \\ \text{its algebraic counterpart } \mathrm{Alg}(\mathcal{L}) \text{ is a finitely based quasi-variety,}\end{gathered}$$

where a "*finitely based (quasi-)variety*", by definition, is a class of algebras which can be axiomatized by a finite set of (quasi-)equations.

The classical example for a logic, where the statement on the left hand side of the above double implication is satisfied (and thus also the one on the right hand side of it), is classical propositional logic. Here we have a Hilbert style inference system with a truly finite schema of logical axioms on one hand; and, on the other hand, the corresponding class, BA is a finitely based variety (by Stone's theorem).

If we consider FOL, the picture is different. The axioms and rules of the usual inference system of FOL refer to individual variables, and thus they do not form a finite schema. Looking at the other side of the implication, we find that, though the natural algebraic counterpart RCA_ω of FOL is a variety (i.e., axiomatizable by a set of equations), it is very far from being axiomatizable by a finite set of equations. Actually, any set Σ of equations axiomatizing RCA_ω must contain infinitely many variables, moreover, to any $n \in \omega$, there is an $e \in \Sigma$ containing more than n fundamental operations as well as more than n variables, a result of Andréka (improving a result of Monk), cf. [27], [1].

2.3 Finitization problem

1) The finitization problem asks whether the negative result quoted in the last paragraph of subsection 2.2 is an inherent property of FOL or not. Perhaps the negative result is due to an unfortunate but not necessary choice of the basic logical connectives (the Boolean connectives plus quantifiers and equalities). Perhaps one could choose the basic logical connectives in such a way that the above negative property of the derivation system of FOL, or equivalently, the so complex non-finite axiomatizability of the algebraic counterpart of FOL would disappear.

This problem (and many variants) was formulated several times in the literature of logic and algebraic logic. We mention here only some of them: [24], [14, Problem 1], [27]. In [24], J. D. Monk formulates the finitization problem as follows:

> *"Devise an algebraic version of predicate logic in which the class of representable algebras forms a finitely based equational class".*

2) A strongly related problem is raised and discussed in [40, pp.56–62]. The research reported in [40] is aiming at finitizing set theory — finitizing this way the whole of mathematics, in that the traditional foundation of mathematics is set theory. This problem, viewed from the perspective of FOL, has to do with finitizing (algebraic counterparts of) *finite variable fragments* of FOL, cf. e.g. [28], [37], [2, p.16], while the former problem deals with usual FOL having a sequence of variables of order type ω. In other words, in [40], instead of the expressive power of full FOL, they only require that of the three–variable fragment of FOL to be still available in the new formulation.

Alfred Tarski formulated set theory in the equational theory of *relation algebras*. The classes of algebras (QRA, TPA, and their variants) investigated in the present paper were introduced first in [40].

There are some further natural requirements, namely that all logical connectives as well as their algebraic counterparts should be permutation invariant, cf. [27], [35], [40, §3.5, p. 57 item 3.5(i)], Jónsson [17], [8, p. 55 lines 16–20]. We note that this requirement is needed, for FOL, to fulfill a most natural model theoretic requirement, namely, that isomorphic models satisfy the same formulas. A discussion related to the necessity of permutation invariance is [39].

The present paper addresses directly the second problem, stressing that we require permutation invariance briefly described in the previous paragraph.

We note that [37] gives a good exposition of the finitization problem. Some recent results concerning the finitization problem are mentioned in our subsection 7.

2.4 Finitization of first order logic *with* equality

As we mentioned in the Introduction, the finitization problem for FOL without equality does have a solution. Now we discuss an approach attacking the finitization problem for FOL with equality which leads to the use of TPA's. When doing this, below we outline further facts supporting Thompson's conjecture (cf. in the Introduction) that the finitization problem might *not* admit a positive solution for FOL with equality. This is in sharp contrast with the case of logic without equality. In this connection, we feel, algebraic logic might give some useful information back to logic.

This approach seems to be similar to that of Craig–Vaught [9], and was suggested in the lecture [20] and independently (in different form) in Veloso–Haeberer [42], [43]. It is based on Tarski's **q***uasi-projective* **r***elation* **a***lgebras* (QRA's) (see [40], to be recalled in section 3 below). The idea is that QRA is a reduct of a finitely based variety and all QRA's are representable. As it was pointed out by Leon Henkin, George McNulty, and other participants during the discussion of the lecture [20], three problems have to be solved in trying to implement this idea. These are the following.

1. QRA is *not* a variety, and the finitization problem, cf. e.g. Henkin–Monk [14], explicitly writes "...class of representable algebras forms a finitely based *equational class*". So the problem asks for a variety and not for a reduct of one.

2. The projection functions are not logical in the sense of [40], i.e., they are not permutation invariant in Jónsson's sense.

3. The set QRA's have all infinite bases, which seems to mean that the corresponding logic has no finite models (at least if we use the standard translation between algebraic logic and logic as described in our subsection 2.1).

To alleviate item 1, we could add two constant symbols say p and q to the language of QRA, and add some axioms on these constants (like $p^{-1} \circ q = 1$). The intended meaning is that p and q are the so called *quasi-projections*. The new finitely based variety QRA^+ is such that its RA-reduct is exactly QRA. So QRA^+ is a candidate for being the variety the Henkin–Monk problem asks for. The problem requires that every element of QRA^+ should be *representable*. Representability of an algebra $\mathfrak{A}$ means that $\mathfrak{A}$ is isomorphic to some $\mathfrak{A}^+$ *all* operations of which are *concrete* set theoretic ones (like intersection, complementation, relation composition etc.). Therefore the constant operations p and q of $\mathfrak{A}^+$ should be concrete set theoretic constants (like the empty set, the identity relation etc.). Briefly, we say that p and q of QRA^+ should be representable, too. As it was anticipated by Henkin ([13]), we will see that this is not easy to arrange. Of course, we may add finitely many new equations to those defining QRA^+ to ensure representability of p and q (of some subvariety V of QRA^+ with V sufficiently large for the original purposes). Theorem 3.4 below seems to say that this is still not easy to do. In connection with representability of members of QRA^+ we note that, e.g. in Definition 3.1 below, the subclass TPA of QRA^+ consists of representable algebras because p and q are represented as concrete set theoretic constants in the definition of TPA. For the whole of QRA^+, the present authors have not yet seen a proposed notion of representation that would intend to be similar in concreteness to the representation proposed by TPA. Summing up, QRA^+ is a finitely based variety, but we do not see a representation for it. Therefore, we turn to its subclass $\mathbf{SP}(\mathsf{TPA})$ consisting of representable algebras to see if we can obtain a finitely based variety from that. Note that our problem in item 1 above does *not* involve the question of whether the operations are logical. So if we investigate the Henkin–Monk problem without the new requirement added in [40], our problem 1 above (i.e., Theorem 3.4 below) still has to be dealt with somehow.

Let us return to a more concrete investigation of problems 1–3 above. [20] proposed to approach this problem by replacing the usual set QRA's with TPA's. Independently, the same approach is taken in [42], [43]. Roughly, a TPA is a set QRA with a base U and a distinguished subset U_0 of U such that the elements of U_0 are not pairs while U is the closure of U_0 under forming ordered pairs. That is

(1) $U = \bigcup_{n\in\omega} U_n, \quad \text{where } U_{n+1} = (U_n \times U_n) \cup U_n.$

Further, the projection functions of this QRA are the standard set theoretic ones. Then we say that U_0 corresponds to the universe of that model (of logic) from which our TPA is obtained and the rest of U, i.e., $U \smallsetminus U_0$ corresponds to (part of the) logic itself. Hence the elements of $U \smallsetminus U_0$ are "logical" elements. Therefore, when this approach reaches the point where the notion of "set algebra" or representable algebra is introduced, then the elements of $U \smallsetminus U_0$ should be required to be some "standard set theoretical" constructs (obtained from U_0). Indeed, this is the spirit in which TPA's are defined in [20], [21], [42], [43]. Actually, in [42], [43] the word is not "TPA" but "SECBR" (cf. the relevant paragraph in the Introduction), but the results and methods of the present paper apply to those algebras the same way as to TPA's, see [43, §3]. From the point of view of the present paper it is interesting to note that, around the end of [43, §6 (comparison of TPA's and cylindric algebras)], they seem to conjecture that the equational theory of TPA might be finitely axiomatizable. We will prove below that no expansion of the theory can have the conjectured property.

3 Complexity of $\mathrm{Eq}(\mathsf{TPA})$ and some consequences

Let us recall from e.g. [15, Def.5.3.1 p.211] that a r*elation* a*lgebra* (RA for short) is a system $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \circ, {}^{-1}, 1'\rangle$ such that $\langle A, +, \cdot, -, 0, 1\rangle$ is a BA, $\circ$ is a binary operation on A (called composition), ${}^{-1}$ is a unary operation on A (called converse), $1' \in A$ (called identity), and $\mathfrak{A}$ satisfies certain conditions. Recall, e.g. from Tarski–Givant [40], that the class of *quasi-projective relation algebras* is

$$\mathsf{QRA} \stackrel{\text{def}}{=} \{\mathfrak{A} \in \mathsf{RA} : \mathfrak{A} \models \exists x \exists y ((x^{-1} \circ x + y^{-1} \circ y \leq 1') \wedge (x^{-1} \circ y = 1))\}.$$

When speaking of a QRA $\mathfrak{A}$, its elements x, y in the above formula are called $\mathfrak{A}$'s *projection functions*, and are denoted by p and q. By a *set* QRA we understand one that is a relation set algebra in the sense of [15]. $\mathfrak{Re}(U)$ denotes the full relation set algebra with base U in the sense of [15, Def. 5.3.2]. Thus the greatest element of $\mathfrak{Re}(U)$ is $U \times U$ and all its elements are binary relations on U.

3.1 DEFINITION ([20], [21, §4 Def. 20])

1. $\mathfrak{A}$ is called a *full true pairing algebra* (full TPA for short) iff $\mathfrak{A} = \langle \mathfrak{R}, p, q \rangle$, where $\mathfrak{R} = \mathfrak{Re}(U)$ for some set U and

$$p = \{\langle\langle x, y\rangle, x\rangle : x, y \in U\}, \quad q = \{\langle\langle x, y\rangle, y\rangle : x, y \in U\}.$$

   p and q are called *projection functions* of $\mathfrak{A}$.

2. $\mathsf{TPA} = \mathbf{SP}(\text{full } \mathsf{TPA})$, i.e., $\mathfrak{A} \in \mathsf{TPA}$ iff $\mathfrak{A}$ is isomorphic to a subalgebra of a direct product of full TPA's. TPA is called the class of all *true pairing algebras*.

$\triangle$

In formulating Theorem 3.2 below, we use the terminology of recursion theory for clarifying the "uncomputability" of non-computable functions. The complexity classes are denoted by Π^k_n and Σ^k_n $(n, k \in \omega)$, where if $k > 0$, then the set in question is so far from being computable that it is called non-arithmetical. (This implies that such sets are not definable in set theory without the Axiom of Infinity.)

3.2 THEOREM *The set* $\mathrm{Eq}(\mathsf{TPA})$ *of all equations valid in* TPA *is* Π^1_1 *complete (in the analitical hierarchy). That is, the "degree of unsolvability" or measure of definability of* $\mathrm{Eq}(\mathsf{TPA})$ *is exactly* Π^1_1.

Proof. This is an immediate consequence of Theorems 4.1 and 5.1 below. $\square$

3.3 COROLLARY *The equational theory of* SECBR *is* Π^1_1*-complete.*

Proof. The TPA-operations are term-definable in SECBR, hence our theorem immediately yields this corollary. $\square$

3.4 THEOREM *1. The variety generated by* TPA *is not finitely based (i.e.,* $\mathbf{HSP}(\mathsf{TPA})$ *is not finitely axiomatizable). Moreover,* $\mathbf{HSP}(\mathsf{TPA})$ *is not axiomatizable by any decidable (or even recursively enumerable) set of axioms.*

2. *The first order theory* $\mathrm{Th}(\mathsf{TPA})$ *of* TPA *is not finitely axiomatizable.* $\mathrm{Th}(\mathsf{TPA})$ *is not axiomatizable by any decidable set either.*

Proof. This is an immediate consequence of Theorem 3.2, but, for completeness, we give a more direct proof too.

(1) Assume that $\mathbf{HSP}(\mathsf{TPA}) = \mathrm{Mod}(E)$ where E is a recursively enumerable (r.e.) set of equations. Then $\{e \in \text{Equations} : \mathsf{TPA} \models e\} = \{e \in \text{Equations} : E \vdash e\}$ is r.e., contradicting an immediate consequence of Theorem 4.1 below: since $\mathrm{Eq}(\mathsf{TPA})$, i.e., all equations valid in TPA, is at least on the Π^1_1 level by Theorem 4.1, we get that $\mathrm{Eq}(\mathsf{TPA})$ is not r.e. Since every finite set is r.e., this completes the proof of (1).

(2) Assume that $\mathrm{Th}(\mathsf{TPA}) = \{\phi : Ax \vdash \phi\}$ with Ax r.e. Then $\{e \in \text{Equations} : e \in \mathrm{Th}(\mathsf{TPA})\}$ is r.e., too, contradicting the above fact again. $\square$

4 The complexity of $\mathrm{Eq}(\mathsf{TPA})$ is at least Π^1_1

Let $\Omega = \langle \omega, \mathrm{pred}, +, \cdot \rangle$, where pred, $+, \cdot$ are, respectively, the usual predecessor, addition, and multiplication on the set ω of natural numbers. Thus

$$\mathrm{pred} = \{\langle n, n-1 \rangle : 0 < n \in \omega\}.$$

4.1 Theorem *Let ψ be a Π^1_1 sentence of arithmetic. Then there is an equation ϵ_ψ in the language of TPA, effectively calculable from ψ, such that*

$$\Omega \models \psi \quad\Longleftrightarrow\quad \mathsf{TPA} \models \epsilon_\psi.$$

Proof. By a <u>*standard pairing structure*</u> we understand a structure $\langle U, p, q \rangle$ where U is as in (1) at the end of the previous section, and p, q are the two projection functions, see in item 1 of Def.3.1 above.

First we write up a quasi-equation $(e_1 \wedge e_2 \wedge e_3)(N, A, M)$ in the language of TPA containing three variables N, A, M. The intended meaning of $(e_1 \wedge e_2 \wedge e_3)(N, A, M)$ is that N is the set of <u>n</u>atural numbers and $A : N \times N \longrightarrow N$, $M : N \times N \longrightarrow N$ are the functions <u>a</u>ddition and <u>m</u>ultiplication, respectively. (All this is meant up to isomorphism.) The first order version $\Phi_N \wedge \Phi_A \wedge \Phi_M$ of $e_1 \wedge e_2 \wedge e_3$ contains

$$\begin{aligned}\Phi_N \overset{\text{def}}{=} (\forall x \in N)(\exists y \in N)(p(y) = x = q(y)) \wedge |N \smallsetminus \mathrm{Dom}(p)| = 1 \\ \wedge (\forall x \in N \cap \mathrm{Dom}(p))(p(x) = q(x) \in N).\end{aligned}$$

If $\langle U, p, q \rangle$ is a standard pairing structure and $\langle U, p, q, N \rangle \models \Phi_N$, then $\langle N, p \restriction N \rangle \cong \langle \omega, \mathrm{pred} \rangle$. This is true because by the Axiom of Foundation (in the set theory we are working in) $(\forall x \in N)(\exists n \in \omega)(p^n(x) \notin \mathrm{Dom}(p)$ but $p^n(x)$ is defined). Then letting $f(x) \overset{\text{def}}{=} n$, we defined a bijection $f : N \rightarrowtail\!\!\!\!\twoheadrightarrow \omega$ which turns out to be the desired isomorphism.

In describing the equational forms (of Φ_N etc.), we will need the following fact. It is well known in RA-theory that any Boolean combination of equations is equivalent with a single equation in the subdirectly irreducible RA's. This follows, e.g., from the fact that RA's form a discriminator variety. For a careful and illuminating exposition of all this see [16]. (We note that in algebraic logic this property of RA's was known already in the 19th century.)

The equational form of Φ_N is (with N as a variable)

$$e_1 \stackrel{\text{def}}{=} (1.1) \wedge (1.2) \wedge (1.3) \wedge (1.4) \wedge (1.5), \text{ where}$$

(1.1) $\stackrel{\text{def}}{=}$ $(N \leq 1')$, meaning, of course, that $N \subseteq \{\langle x, x\rangle : x \in U\}$

(1.2) $\stackrel{\text{def}}{=}$ $(1 \circ N \circ (p \cap q) \geq N)$, i.e., $\{\langle x, y\rangle : \exists z(\langle z, \langle y, y\rangle\rangle \in N\} \supseteq N$,
which, together with (1.1), implies that N is closed under successor

(1.3) $\stackrel{\text{def}}{=}$ $(1 \circ N \circ 1 = 1)$, i.e., $N \neq \emptyset$

(1.4) $\stackrel{\text{def}}{=}$ $[1 \circ (N - (p \circ 1))] \cap [(N - (p \circ 1)) \circ 1] \leq 1'$, i.e., $|(U_0 \times U_0) \cap N| \leq 1$,
uniqueness of 0

(1.5) $\stackrel{\text{def}}{=}$ $(N \circ p = N \circ q \leq 1 \circ N)$, i.e.,
$(\forall x \in N \cap \mathrm{Dom}(p) \cap \mathrm{Dom}(q))(p(x) = q(x) \in N)$,
i.e., N is closed under predecessor.

(*1) If $\mathfrak{B} = \langle \mathfrak{Re}(U), p, q\rangle \in \mathsf{TPA}$ and $N \subseteq U$, then $\mathfrak{B} \models e_1[\mathrm{Id} \restriction N]$ implies that $\langle U, p, q, N\rangle \models \Phi_N$ hence $\langle N, p\rangle \cong \langle \omega, \mathrm{pred}\rangle$.

By (*1) we manage to "code" the set of natural numbers by an equation e_1 in TPA's. This is similar to the beginning of the proof of Thm.2 in Németi [26], see the explanation of the idea of the proof of Thm.3.1 on p.252 therein, or the following picture of N (described by e_1).

$$\begin{array}{c} \vdots \\ \langle\langle\langle u_0, u_0\rangle, \langle u_0, u_0\rangle\rangle, \langle\langle u_0, u_0\rangle, \langle u_0, u_0\rangle\rangle\rangle \\ \downarrow p \\ \langle\langle u_0, u_0\rangle, \langle u_0, u_0\rangle\rangle \\ \downarrow p \\ \langle u_0, u_0\rangle \end{array}$$

Here $u_0 \in U_0$. Note that $\mathrm{Dom}(N)(= \mathrm{Rng}(N))$ satisfies Φ_N in $\langle U, p, q, N\rangle$.

The rest of our proof will proceed analogously to that proof of Németi [26], too. (More precisely, the first order part of this proof is analogous to the one in [26]. Lifting this first order part to Π^1_1 seems to involve new ideas.) We are writing up formulas in the language of the structure $\mathfrak{A} = \langle U, p, q, N, A, M\rangle$, where $A, M \subseteq U \times U$ are partial unary operations (and the rest are as above). To define Φ_A, we introduce the abbreviation

$$+(v_0, v_1, v_2) \overset{\text{def}}{\Longleftrightarrow} \exists x[(p(x) = v_0 \wedge q(x) = v_1 \wedge A(x) = v_2].$$

Now

$$\Phi_A \overset{\text{def}}{=} (A.1) \wedge (A.2) \wedge (A.3) \wedge (A.4) \wedge (A.5) \wedge (A.6),$$

where

(A.1) $\overset{\text{def}}{=}$ $\forall v(p(v), q(v) \in N \to A(v) \in N)$

(A.2) $\overset{\text{def}}{=}$ $\forall v(\forall x, y, z \in N)(+(x, y, z) \wedge +(x, y, v) \to z = v)$

(A.3) $\overset{\text{def}}{=}$ $(\forall x, y, z \in N)(+(x, y, z) \leftrightarrow +(y, x, z))$

(A.4) $\overset{\text{def}}{=}$ $(\forall x, y, z \in N)(+(x, p(y), p(z)) \to +(x, y, z))$

(A.5) $\overset{\text{def}}{=}$ $(\forall x, y \in N)(y \notin \mathrm{Dom}(p) \leftrightarrow +(x, y, x))$

(A.6) $\overset{\text{def}}{=}$ $\forall v(v \in \mathrm{Dom}(A) \leftrightarrow v \in \mathrm{Dom}(M) \leftrightarrow (p(v) \in N \wedge q(v) \in N))$.

(*2) As in the case of Φ_N,if $\langle U, p, q\rangle$ is a standard pairing structure, then

$$\mathfrak{A} \overset{\text{def}}{=} \langle U, p, q, N, A, M\rangle \models \Phi_N \wedge \Phi_A \text{ implies } \left\langle N, p \restriction N, +^U \right\rangle \cong \langle \omega, \mathrm{pred}, +\rangle,$$

where

$$+^U \overset{\text{def}}{=} \{\langle a, b, c\rangle : \langle a, b\rangle \in \mathrm{Dom}(A) \text{ and } A(\langle a, b\rangle) = c\}.$$

The equational form of Φ_A is (N, A and M are variables again)

$$e_2(N, A, M) \overset{\text{def}}{=} (2.1) \wedge (2.2) \wedge (2.3) \wedge (2.4) \wedge (2.5) \wedge (2.6),$$

where (2.1), ..., (2.6) correspond to $(A.1), \ldots, (A.6)$, respectively.

(2.1) $\overset{\text{def}}{=}$ $(q \circ (p \cap q) \circ N \circ 1) \cap (p \circ (p \cap q) \circ N \circ 1) \leq A \circ (p \cap q) \circ N \circ (p \cap q)^{-1}$

(2.2) $\overset{\text{def}}{=}$ $C^{-1} \circ C \leq 1'$, where $C \overset{\text{def}}{=} A \cap (q \circ (p \cap q) \circ N \circ 1) \cap (p \circ (p \cap q) \circ N \circ 1)$

(2.3) $\overset{\text{def}}{=}$ $C^{-1} \circ ((p \circ q^{-1}) \cap (q \circ p^{-1})) \circ C \leq 1'$

(2.4) $\overset{\text{def}}{=}$ $\big((p \circ (p \cap q) \circ N \circ (p \cap q)^{-1} \circ p^{-1}) \cap (q \circ (p \cap q) \circ N \circ q^{-1})\big) \circ$
$A \circ (p \cap q)^{-1} \leq C$

(2.5) $\stackrel{\text{def}}{=}$ $(p \circ (p \cap q) \circ N \circ (p \cap q)^{-1}) \cap (q \circ D \circ 1) = (C \cap p)$,
where $D \stackrel{\text{def}}{=} [1 \circ (N - (p \circ 1))] \cap [(N - (p \circ 1) \circ 1]$

(2.6) $\stackrel{\text{def}}{=}$ $A \circ 1 = M \circ 1 = (p \circ (p \cap q) \circ N \circ 1) \cap (q \circ (p \cap q) \circ N \circ 1)$.

Now, exactly as in the case of Φ_N, $\langle \mathfrak{Re}(U), p, q \rangle \models (e_1 \wedge e_2)[\mathrm{Id} \restriction N, A, M]$ iff $\mathfrak{A} \models \Phi_N \wedge \Phi_A$, where $\mathfrak{A}$ is as in (*2), and $(e_1 \wedge e_2)[\mathrm{Id} \restriction N, A, M]$ is an evaluated equation, i.e., variables are replaced by concrete elements of the TPA in question.

Let

$$m(x, y, z) \stackrel{\text{def}}{\iff} (\exists v)(p(v) = x \wedge q(v) = y \wedge M(v) = z).$$

To ensure that m is real multiplication, we let (analogously to the definition of Φ_A) Φ_M be the formula $(M.1) \wedge \ldots \wedge (M.5)$, where

(M.1) $\stackrel{\text{def}}{=}$ $M : N \times N \longrightarrow N$ is onto

(M.2) $\stackrel{\text{def}}{=}$ $\forall v(\forall x, y, z \in N)(m(x, y, z) \wedge m(x, y, v) \to z = v)$

(M.3) $\stackrel{\text{def}}{=}$ $(\forall x, y \in N)(x \notin \mathrm{Dom}(p) \leftrightarrow m(x, y, x))$

(M.4) $\stackrel{\text{def}}{=}$ $(\forall x, y, z \in N)(m(x, y, z) \leftrightarrow m(y, x, z))$

(M.5) $\stackrel{\text{def}}{=}$ $(\forall x, y, z \in N)(m(p(x), y, z) \to (\exists w)(+(z, y, w) \wedge m(x, y, w)))$.

If $\mathfrak{A}$ is as in (*2), then

(*3) $\mathfrak{A} \models \Phi_N \wedge \Phi_A \wedge \Phi_M$ implies $\left\langle N, p \restriction N, +^U, m^U \right\rangle \cong \langle \omega, \mathrm{pred}, +, \cdot \rangle$, if $\langle U, p, q \rangle$ is a standard pairing structure.

$(M.4)$ can be divided into two parts $(M.4a)$ and $(M.4b)$ in the following way.

$$(\forall x, y, z \in N)(m(x, y, z) \leftrightarrow m(y, x, z)) \iff$$
$$\forall v(\forall x, y, z \in N)(m(x, y, z) \wedge m(y, x, v) \to z = v) \wedge$$
$$\forall v(\forall x, y, z \in N)(m(x, y, z) \leftrightarrow m(y, x, v)).$$

The equational form of Φ_M is $e_3(N, A, M) \stackrel{\text{def}}{=} (3.1) \wedge \ldots \wedge (3.4a) \wedge (3.4b) \wedge (3.5)$, where $(3.1), \ldots, (3.5)$ correspond to $(M.1), \ldots, (M.5)$, respectively.

(3.1) $\stackrel{\text{def}}{=}$ $M^{-1} \circ \big((p \circ (p \cap q) \circ N \circ 1) \cap (q \circ (p \cap q) \circ N \circ 1)\big) = (p \cap q) \circ N \circ 1$

(3.2) $\stackrel{\text{def}}{=}$ $(p \cap q) \circ N \circ (p \cap q)^{-1} \circ K \circ K^{-1} \leq 1'$,
where $K \stackrel{\text{def}}{=} \big((p \circ (p \cap q) \circ N \circ 1) \cap (q \circ (p \cap q) \circ N \circ 1)\big) \cap M$

(3.3) $\stackrel{\text{def}}{=}$ $(p \circ D) \cap (q \circ (p \cap q) \circ N \circ 1) = p \cap K$

(3.4a) $\stackrel{\text{def}}{=}$ $(p \cap q) \circ N \circ (p \cap q)^{-1} \circ K^{-1} \circ ((p \circ q^{-1}) \cap (q \circ p^{-1})) \circ K \leq 1'$

(3.4b) $\stackrel{\text{def}}{=}$ $((p \circ q^{-1}) \cap (q \circ p^{-1})) \circ K \circ (p \cap q) \circ N \circ (p \cap q)^{-1} \leq K$

(3.5) $\stackrel{\text{def}}{=}$ $J \leq ((J \circ p^{-1}) \cap (q \circ q^{-1})) \circ [((C \circ q^{-1}) \cap (p \circ p^{-1})) \cap$
$[((p \circ C^{-1}) \cap (q \circ q^{-1})) \circ ((p \circ (p \cap q)^{-1} \circ p^{-1}) \cap (q \circ q^{-1}) \circ M \circ q^{-1})] \circ p$,
where $J \stackrel{\text{def}}{=} ((p \circ (p \cap q) \circ N \circ p^{-1}) \cap (q \circ (p \cap q) \circ N \circ (p \cap q)^{-1} \circ q^{-1})) \circ$
$M \circ (p \cap q) \circ N \circ (p \cap q)^{-1}$, i.e., J denotes those pairs $\langle \langle x, y \rangle, z \rangle$
for which $x, y, z \in N$ and $m(p(x), y, z)$.

Exactly as in the case of e_2, this equation $e_3(N, A, M)$ is such that

$$\langle \mathfrak{Re}(U), p, q \rangle \models (e_1 \wedge e_2 \wedge e_3)[\mathrm{Id} \restriction N, A, M] \text{ iff } \mathfrak{A} \models \Phi_N \wedge \Phi_A \wedge \Phi_M.$$

Let $\mathfrak{B} \subseteq \langle \mathfrak{Re}(U), p, q \rangle$ be any TPA with $X, A, M \in B$ such that $\mathfrak{B} \models (e_1 \wedge e_2 \wedge e_3)[X, A, M]$. Let $N = \mathrm{Dom}(X)$. Then $X = \mathrm{Id} \restriction N$ and hence $\langle \Re(U), p, q \rangle \models (e_1 \wedge e_2 \wedge e_3)[\mathrm{Id} \restriction N, A, M]$, thus $\langle U, p, q, N, A, M \rangle \models \Phi_N \wedge \Phi_A \wedge \Phi_M$ proving

(*4) $\langle N, p \restriction N, +^U, m^U \rangle \cong \Omega$

by (*3) and the above properties of $e_1 \wedge e_2 \wedge e_3$. Summing up, we have (*5) below, where baseof($\mathfrak{B}$) denotes the set U for which $\mathfrak{B} = \langle \mathfrak{Re}(U), \ldots \rangle$.

(*5) For any TPA $\mathfrak{B}$ and $X, A, M \in B$ we have that if $\mathfrak{B} \models e_1 \wedge e_2 \wedge e_3[X, A, M]$ then $\langle \mathrm{baseof}(\mathfrak{B}), p^{\mathfrak{B}}, q^{\mathfrak{B}}, \mathrm{Dom}(X), A, M \rangle \models \Phi_N \wedge \Phi_A \wedge \Phi_M$, so $\mathrm{Dom}(X), p^{\mathfrak{B}}, A, M$ code an isomorphic copy of the standard model Ω of arithmetic as indicated in (*4).

When applying (*5), we will need (5.1) below known from the literature. Let $\langle U, p, q \rangle$ be a standard pairing structure and $X, A, M, P^+ \in \mathcal{P}(U \times U)$ with $X \cup P^+ \subseteq \mathrm{Id}$. Let $N = \mathrm{Dom}(X)$ and $P = \mathrm{Dom}(P^+)$ and ϕ be a first order formula in the language of $\langle U, p, q, N, A, M, P \rangle$. Now we claim that

(5.1) there is a recursive function associating to every first order formula ϕ of the above kind a TPA-equation $e_\phi(X, A, M, P^+)$ such that for every $N, P \subseteq U$ and $A, M \subseteq U \times U$,

$$\langle \mathfrak{Re}(U), p, q \rangle \models e_\phi[\mathrm{Id} \restriction N, A, M, \mathrm{Id} \restriction P] \text{ iff } \langle U, p, q, N, A, M, P \rangle \models \phi.$$

The above (5.1) is proved in [40, §4.5], see the theorem on p.126 (there it is stated in a different form, see also §7.1) as well as in [25, Lemma 3, p.35] (where the result is stated in the present form; our e_ϕ is denoted there as $g(\phi) = 1$ because $g(\phi)$ is a TPA-term there).

It is well known that every statement of arithmetic of the form $\forall P_1 \ldots \forall P_n \forall F_1 \ldots \forall F_m \phi$ can be equivalently expressed by a formula of the form

$\forall P\psi$, where both ϕ and ψ are first order formulas, $P_1, \ldots, P_n, P$ are predicate variables, and $F_1, \ldots, F_m$ are function variables. (If $F_i : {}^kN \longrightarrow N$, then this F_i can be replaced by a predicate variable which has $k+1$ arguments. See Rogers [29, §16.1 Thm.III] for the rest.) Moreover, P can be replaced by a one argument predicate variable. After this reduction, for any Π^1_1 sentence ψ (of the form $\forall P\phi$, where ϕ is first order) of the language of Ω we construct ψ^+ ($= \forall P\phi^+$, where ϕ^+ is first order again) in the language of $\langle U, p, q, N, A, M\rangle$ such that

(*6) $\Omega \models \psi$ iff $\langle U, p, q, N, A, M\rangle \models (\Phi_N \wedge \Phi_A \wedge \Phi_M \longrightarrow \psi^+)$, for any standard pairing structure $\langle U, p, q\rangle$ and any choice of N, A, M.

Indeed, it is easy to see that ψ^+ can be defined by induction on the complexity of ψ, e.g., let $(i = j + k)^+$ be $\exists v(A(v) = i \wedge p(v) = j \wedge q(v) = k)$ ... and write $p(v)$ for $\mathrm{pred}(v)$. Now apply (*3).

Then, by (*5) and (*6) we have that

$$\Omega \models \psi \quad \text{iff}$$

(*7)
$$(\forall \mathfrak{B} \in \mathsf{TPA})(\forall X, A, M \in B)(\mathfrak{B} \models (e_1 \wedge e_2 \wedge e_3)[X, A, M] \Rightarrow$$
$$\big\langle \mathrm{baseof}(\mathfrak{B}), p^{\mathfrak{B}}, q^{\mathfrak{B}}, \mathrm{Dom}(X), A, M\big\rangle \models \psi^+).$$

As each possible value of P (in the formula $\forall P\ \phi$) is a subset of ω, to $\psi^+ = \forall P\phi^+$ we associate a quasi-equation $e_\psi = (P^+ \leq X \wedge X \leq \mathrm{Id}) \to e_\phi$, where e_ϕ (or, more precisely, e_{ϕ^+}) is the TPA-equation corresponding to ϕ^+ according to (5.1) above. Let $\mathfrak{B}, X, A, M$ be as in (*7). Since ψ^+ is $\forall P\phi^+$, we have

$$\big\langle \mathrm{baseof}(\mathfrak{B}), p^{\mathfrak{B}}, q^{\mathfrak{B}}, \mathrm{Dom}(X), A, M\big\rangle \models \psi^+) \text{ iff}$$
$$(\forall P \leq \mathrm{Dom}(X))\,\big\langle \mathrm{baseof}(\mathfrak{B}), p^{\mathfrak{B}}, q^{\mathfrak{B}}, \mathrm{Dom}(X), A, M, P\big\rangle \models \phi^+).$$

By (5.1), the right hand side holds iff for our equation e_ϕ (associated to ϕ^+)

$$\mathfrak{B} \models (P^+ \leq X \to e_\phi[X, A, M, \mathrm{Id} \restriction P],$$

which is equivalent to

$$\mathfrak{B} \models \forall P^+ e_\psi[X, A, M].$$

So we have $(\forall \mathfrak{B} \in \mathsf{TPA})(\forall X, A, M \in B)$

$$\big(\big\langle \mathrm{baseof}(\mathfrak{B}), p^{\mathfrak{B}}, q^{\mathfrak{B}}, \mathrm{Dom}(X), A, M\big\rangle \models \psi^+ \text{ iff } \mathfrak{B} \models \forall P^+ e_\psi[X, A, M]\big).$$

Putting this together with (*7), we get that

$$\Omega \models \psi \quad \text{iff}$$
$$(\forall \mathfrak{B} \in \mathsf{TPA})(\forall X, A, M, P^+ \in B)(\mathfrak{B} \models (e_1 \wedge e_2 \wedge e_3)[X, A, M] \Rightarrow$$
$$\mathfrak{B} \models e_\psi[X, A, M, P^+]).$$

But this means exactly that

(*8) $\Omega \models \psi$ iff $\mathsf{TPA} \models (e_1 \wedge e_2 \wedge e_3)(X, A, M) \rightarrow e_\psi(X, A, M, P^+)$.

Now, by the above result of Jónsson, there is an equation e^+_ψ such that $\mathsf{TPA} \models (e_1 \wedge e_2 \wedge e_3) \rightarrow e_\psi$ iff $\mathsf{TPA} \models e^+_\psi$ and the function $\{\langle e_\psi, e^+_\psi\rangle : \psi \text{ is a formula in the language of } \Omega\}$ is computable. So, by (*8), we have

(*9) $\Omega \models \psi$ iff $\mathsf{TPA} \models e^+_\psi$

and there is a computable function f with $e^+_\psi = f(\psi)$ for any formula ψ in the Π^1_1 language of Ω. This f is computable because Lemma 3 of Németi [25] explicitly says so (and the rest of our above construction, including Jónsson's, are clearly computable). Thus we have

$$\langle \omega, \mathrm{pred}, +, \cdot\rangle \models \psi \Longleftrightarrow \mathsf{TPA} \models \epsilon_\psi$$

which completes our proof. $\square$

5 The complexity of $\mathrm{Eq}(\mathsf{TPA})$ is at most Π^1_1

The following theorem states that $\mathrm{Eq}(\mathsf{TPA})$ is at most on the Π^1_1 level, and so, by Theorem 4.1, it is exactly on the Π^1_1 level.

5.1 Theorem *Let e be an equation in the language of* TPA*'s. Then there is a Π^1_1 sentence ψ_e (effectively calculable from e) such that*

*(*10)* $\quad \mathsf{TPA} \models e \quad \Longleftrightarrow \quad \Omega \models \psi_e.$

Proof. We will use the well known and convenient λ-notation to form expressions of the language of arithmetic. So let r be the pairing function $\lambda x \lambda y. 2^x \cdot 3^y$ on ω. From now on, let $\mathbf{U}_0$, $\mathbf{U}$, $\mathbf{K}$, $\mathbf{O}$, $\mathbf{I}$, $\mathbf{P}$, and $\mathbf{Q}$ be predicate (thus second order) variables. We are listing a few formulas that restrict the possible meanings of them so that they can be thought of as representatives of a set of non-pair elements of a TPA, base, (Boolean) identity, (Boolean) zero, multiplicative identity, and projection elements, respectively.

$$\varphi_0 \stackrel{\text{def}}{=} \forall x(\mathbf{U}_0(x) \rightarrow (\neg\exists y \exists z(x = r(y, z))))$$

$$\varphi_1 \stackrel{\text{def}}{=} \forall x(\mathbf{U}(x) \leftrightarrow (\mathbf{U}_0(x) \vee \exists y \exists z(\mathbf{U}(y) \wedge \mathbf{U}(z) \wedge (x = r(y, z)))))$$

$$\varphi_2 \stackrel{\text{def}}{=} \forall x(\mathbf{K}(x) \leftrightarrow \exists y \exists z(\mathbf{U}(y) \wedge \mathbf{U}(z) \wedge (x = r(y, z))))$$

$$\varphi_3 \stackrel{\text{def}}{=} \forall x(\neg\mathbf{O}(x))$$

$$\varphi_4 \stackrel{\text{def}}{=} \forall x(\mathbf{P}(x) \leftrightarrow \exists y \exists z(\mathbf{U}(y) \wedge \mathbf{U}(z) \wedge (x = r(r(y, z), y))))$$

$$\varphi_5 \stackrel{\text{def}}{=} \forall x(\mathbf{Q}(x) \leftrightarrow \exists y \exists z(\mathbf{U}(y) \wedge \mathbf{U}(z) \wedge (x = r(r(y,z),z))))$$

$$\varphi_6 \stackrel{\text{def}}{=} \forall x(\mathbf{I}(x) \leftrightarrow \exists y(\mathbf{U}(y) \wedge (x = r(y,y)))).$$

Assume that $\mathbf{U}_0, \ldots, \mathbf{Q}$ are the first seven predicate variables of the language of arithmetic, the others are enumerated as $\mathbf{V}_1, \ldots, \mathbf{V}_n, \ldots$. If e is $\tau_1 = \tau_2$ then let $k \in \omega$ be such that $\tau_1, \tau_2 \in TRM^k_{\mathsf{TPA}}$ (TPA terms such that, for $i > k$, v_i does not occur in them).

Then ψ_e will be of the form

$$\forall \mathbf{U_0}, \mathbf{U}, \mathbf{K}, \mathbf{O}, \mathbf{I}, \mathbf{P}, \mathbf{Q}, \mathbf{V}_1 \ldots \mathbf{V}_k((\varphi_0 \wedge \cdots \wedge \varphi_6 \\ \wedge \forall x[(\mathbf{V}_1(x) \vee \cdots \vee \mathbf{V}_k(x)) \to \mathbf{K}(x)]) \to (\tau_1^* = \tau_2^*)),$$

where τ_i^* is the one-place predicate of arithmetic obtained from τ_i by applying to it the translating function $*$ given below.

Definition 5.1.1. The translating function $* : TRM^k_{\mathsf{TPA}} \longrightarrow PRED^1_A$ (where $PRED^1_A$ is the set of one place predicates of the language of arithmetic) is defined by the following clauses:

$$1^* = \mathbf{K}, \quad 0^* = \mathbf{O}, \quad 1'^* = \mathbf{I}, \quad p^* = \mathbf{P}, \quad q^* = \mathbf{Q}$$

$$v_i^* = \mathbf{V}_i$$

$$[\sim \tau]^* = \lambda x(\mathbf{K}(x) \wedge \neg \tau^*(x))$$

$$[\tau \cup \mu]^* = \lambda x(\tau^*(x) \vee \mu^*(x))$$

$$[\tau \cap \mu]^* = \lambda x(\tau^*(x) \wedge \mu^*(x))$$

$$[\tau^{-1}]^* = \lambda x \exists y \exists z((x = r(z,y)) \wedge \tau^*(r(y,z)))$$

$$[\tau \circ \mu]^* = \lambda x \exists y \exists z \exists w((x = r(y,w)) \wedge \tau^*(r(y,z)) \wedge \mu^*(r(z,w))).$$

$\triangle$

(Note that for any $\tau \in TRM^k_{\mathsf{TPA}}$, if τ^* is not simply a predicate variable, then it is of the form $\lambda x.\chi(x)$, where χ is a second order formula with no λ's or second order quantifiers in it. Thus $\tau_1^* = \tau_2^*$ is (logically) equivalent to $\forall x(\chi_1 \leftrightarrow \chi_2)$. That is, ψ_e is indeed Π^1_1.) Clearly, for any $\tau \in TRM^k_{\mathsf{TPA}}$ we have

$$\Omega \models \forall \mathbf{U_0}, \mathbf{U}, \mathbf{K}, \mathbf{O}, \mathbf{I}, \mathbf{P}, \mathbf{Q}, \mathbf{V}_1 \ldots \mathbf{V}_k \\ \big((\varphi_0 \wedge \cdots \wedge \varphi_6 \wedge \forall x[(\mathbf{V}_1(x) \vee \cdots \vee \mathbf{V}_k(x)) \to \mathbf{K}(x)]) \to \forall x(\tau^*(x) \to \mathbf{K}(x))\big).$$

Now that the construction of ψ_e is complete, we can start proving (*10).

(⇒) Suppose that $\Omega \not\models \psi_e$, that is, there is an expansion

$$\Omega^+ = \langle \Omega, U_0, U, K, O, I, P, Q, V_1, \ldots, V_k \rangle$$

of Ω such that

$$\Omega^+ \models \varphi_0 \wedge \cdots \wedge \varphi_6 \wedge \forall x \big((\mathbf{V}_1(x) \vee \cdots \vee \mathbf{V}_k(x)) \to \mathbf{K}(x) \big),$$

but

$$(\tau_1^*)^{\Omega^+} \neq (\tau_2^*)^{\Omega^+}.$$

We will construct a TPA

$$\mathfrak{A} = \big\langle \mathcal{P}(U' \times U'), \cup, \cap, \sim, 0, 1, \circ, {}^{-1}, 1', p, q \big\rangle$$

in which the equation e does not hold.

Let U_0' be any set consisting of elements that are not pairs (in the set-theoretical sense), and such that there is a bijection $f' : U_0' \to U_0$. Let U' be the closure of U_0' under forming ordered pairs (as in the first part of the paper), and let f be the $U' \to U$ function defined by

$$f(u) \stackrel{\text{def}}{=} \begin{cases} f'(u), & \text{if } u \in U_0 \\ r(f(u_1), f(u_2)), & \text{if } u = \langle u_1, u_2 \rangle. \end{cases}$$

Then f is one-one and onto because φ_0 and φ_1 are true (in Ω^+). Let $g : U' \times U' \to K$ and $h : \mathcal{P}(U' \times U') \to \mathcal{P}(K)$ be defined by

$$g(\langle u_1, u_2 \rangle) \stackrel{\text{def}}{=} r(f(u_1), f(u_2)) \qquad \text{for all } \langle u_1, u_2 \rangle \in U' \times U',$$

$$h(X) \stackrel{\text{def}}{=} \{g(x) : x \in X\} \qquad \text{for all } X \subseteq U' \times U'.$$

Then g and thus h are one-one and onto because φ_2 is true.

Proposition 5.1.2. If the values of the TPA variables v_i ($i \leq k$) are chosen to be $h^{-1}(V_i)$ (and this makes sense since $h^{-1}(V_i) \subseteq U' \times U'$), then for all $\tau \in TRM^k_{\mathsf{TPA}}$, we have

$$h(\tau^{\mathfrak{A}}) = (\tau^*)^{\Omega^+}.$$

Proof of Prop.5.1.2. By induction on TRM^k_{TPA} using the bijectivity of f, g, and h and the assumption that $\varphi_3, \ldots, \varphi_6$ are true in Ω^+ .

$$h(1^{\mathfrak{A}}) = h(U' \times U') = \{g(x) : x \in U' \times U'\} = (1^*)^{\Omega^+},$$

$$h(0^{\mathfrak{A}}) = h(\emptyset) = \emptyset = (0^*)^{\Omega^+},$$

$$
\begin{aligned}
h(1'^{\mathfrak{A}}) &= \{g(\langle u,u\rangle) : u \in U'\} = \{r(f(u),f(u)) : u \in U'\}\\
&= \{r(x,x) : x \in U\} = I = (1'^{*})^{\Omega^+},
\end{aligned}
$$

$$
\begin{aligned}
h(p^{\mathfrak{A}}) &= \{g(\langle\langle u,v\rangle, u\rangle) : u,v \in U'\}\\
&= \{r(r(f(u),f(v)),f(u)) : u,v \in U'\} = \{r(r(y,z),y) : y,z \in U\}\\
&= P = (p^*)^{\Omega^+}.
\end{aligned}
$$

The proof of $h(q^{\mathfrak{A}}) = (q^*)^{\Omega^+}$ is similar to that of $h(p^{\mathfrak{A}}) = (p^*)^{\Omega^+}$, and $h(v_i^{\mathfrak{A}}) = V_i = (v_i^*)^{\Omega^+}$ by definition.

Suppose that the statement holds for some $\tau,\mu \in TRM^k_{\mathsf{TPA}}$. Then

$$
\begin{aligned}
h([\sim\tau]^{\mathfrak{A}}) &= h(\{v \in U' \times U' : v \notin \tau^{\mathfrak{A}}\})\\
&= \{g(v) : v \in U' \times U' \text{ and } g(v) \notin h(\tau^{\mathfrak{A}})\}\\
&= \{x \in K : x \notin (\tau^*)^{\Omega^+}\} = ([\sim\tau]^*)^{\Omega^+},
\end{aligned}
$$

$$
\begin{aligned}
h([\tau\cup\mu]^{\mathfrak{A}}) &= h(\{v \in U' \times U' : v \in \tau^{\mathfrak{A}} \cup \mu^{\mathfrak{A}}\})\\
&= \{g(v) : g(v) \in (\tau^*)^{\Omega^+} \cup (\mu^*)^{\Omega^+}\}\\
&= \{x : x \in (\tau^*)^{\Omega^A+} \cup (\mu^*)^{\Omega^+}\} = ([\tau\cup\mu]^*)^{\Omega^+},
\end{aligned}
$$

$$
\begin{aligned}
h([\tau^{-1}]^{\mathfrak{A}}) &= h(\{\langle v,u\rangle : \langle u,v\rangle \in \tau^{\mathfrak{A}}\})\\
&= \{r(f(v),f(u)) : r(f(u),f(v)) \in (\tau^*)^{\Omega^+}\}\\
&= \{r(z,y) : r(y,z) \in (\tau^*)^{\Omega^+}\} = ([\tau^{-1}]^*)^{\Omega^+},
\end{aligned}
$$

$$
\begin{aligned}
h([\tau\circ\mu]^{\mathfrak{A}}) &= h(\{\langle u,v\rangle : \exists s(\langle u,s\rangle \in \tau^{\mathfrak{A}} \text{ and } \langle s,v\rangle \in \mu^{\mathfrak{A}})\}\\
&= \{r(f(u),f(v)) : \exists s(r(f(u),f(s)) \in (\tau^*)^{\Omega^+} \text{ and}\\
&\qquad\qquad r(f(s),f(v)) \in (\mu^*)^{\Omega^+})\}\\
&= \{r(x,y) : \exists z(r(x,z) \in (\tau^*)^{\Omega^+} \text{ and } r(z,y) \in (\mu^*)^{\Omega^+})\}\\
&= ([\tau\circ\mu]^*)^{\Omega^+}.
\end{aligned}
$$

This completes the proof of Prop.5.1.2. $\square$

Now, with the valuation (of the TPA-variables) given in the proposition, we have

$$h(\tau_1^{\mathfrak{A}}) = (\tau_1^*)^{\Omega^+} \neq (\tau_2^*)^{\Omega^+} = h(\tau_2^{\mathfrak{A}}),$$

so that $\mathfrak{A} \not\models \tau_1 = \tau_2$, and thus

$$\mathsf{TPA} \not\models \tau_1 = \tau_2.$$

$\underline{(\Leftarrow)}$ The proof relies on the fact that by the choice of the pairing function r there are infinitely many elements that are not "pairs" (i.e., are not elements of $\mathrm{Rng}(r)$).

Suppose that $\mathsf{TPA} \not\models e$. Then (cf. [35]) there is a TPA

$$\mathfrak{A} = \left\langle \mathcal{P}(U' \times U'), \cup, \cap, \sim, 0, 1, {}^{-1}, \circ, 1', p, q \right\rangle$$

such that U' (and thus U'_0, the non-pair elements of U') is countable and e fails in it. Let us fix $\mathfrak{A}$ and define an expansion

$$\Omega^+ = \langle \Omega, U_0, U, K, O, I, P, Q, V_1, \ldots, V_k \rangle$$

of Ω such that

$$\Omega^+ \models \varphi_0 \wedge \cdots \wedge \varphi_6 \wedge \forall x \big((\mathbf{V}_1(x) \vee \cdots \vee \mathbf{V}_k(x)) \to \mathbf{K}(x) \big),$$

but

$$(\tau_1^*)^{\Omega^+} \neq (\tau_2^*)^{\Omega^+}.$$

The pairing function r was defined in such a way that there are infinitely many "no-pairs" in ω (say the powers of 5). Let U_0 be any subset of ω whose cardinality equals that of U'_0 and is such that $U_0 \cap \mathrm{Rng}(r) = \emptyset$. Then

$$\Omega^+ \models \varphi_0.$$

If $f' : U'_0 \to U_0$ is one-one and onto, we define

$$f(u) \stackrel{\text{def}}{=} \begin{cases} f'(u), & \text{if } u \in U'_0 \\ r(f(u_1), f(u_2)), & \text{if } u = \langle u_1, u_2 \rangle,\ u_1, u_2 \in U' \end{cases}$$

and

$$U \stackrel{\text{def}}{=} \mathrm{Rng}(f).$$

Proposition 5.1.3. $f \in {}^{U'}U$, f is one-one and onto.

Proof of Prop.5.1.3. The fact that f is defined for all $u \in U'$ can be proved by induction on the construction of U' (from U'_0). f is clearly onto, so it remains to prove that it is one-one.

Suppose that $f(u) = f(v)$ for some u,$v \in U'$. It is clear that both u and v are either in U'_0 or in $U' \smallsetminus U'_0$, otherwise the image of, say u, would be a "pair", i.e., an element of $\mathrm{Rng}(r)$, but not that of v, contradicting our assumption. If u, $v \in U'_0$ then $u = v$ follows from f's being one-one. Let $u = \langle u_1, u_2 \rangle$, $v = \langle v_1, v_2 \rangle$ be such that $f(u) = f(v)$ is the least $n \in U$ which has more than one pre-image in U'. Since $f(u_1) = f(v_1) < f(u)$ we have $u_1 = v_1$, a contradiction. □

Proposition 5.1.4. $\Omega^+ \models \varphi_1$.

Proof of Prop.5.1.4. ($\to$) If $x \in U$ but $x \notin U_0$, then $x \in \mathrm{Rng}(f) \smallsetminus \mathrm{Rng}(f')$, whence $x = r(f(u), f(v))$ for some u,$v \in U'$, that is $x = r(y, z)$ for some y, $z \in U$.

($\leftarrow$) If $x \in U_0$ then clearly $x \in U$; if $x = r(y, z)$ for some y, $z \in U$ then $x = f(\langle f^{-1}(y), f^{-1}(z)\rangle) \in U$. $\square$

Define g by

$$g(\langle u_1, u_2\rangle) \stackrel{\text{def}}{=} r(f(u_1), f(u_2)) \text{ for all } u_1,\ u_2 \in U'$$

and let

$$K \stackrel{\text{def}}{=} \text{Rng}(g).$$

Then $g : U' \times U' \longrightarrow K$ is one-one and onto.

Proposition 5.1.5. $\Omega^+ \models \varphi_2$.

Proof of Prop.5.1.5. ($\rightarrow$) $x \in K$ implies $x = g(\langle u, v\rangle) = r(f(u), f(v))$ for some u, $v \in U'$. But $f(u)$,$f(v) \in U$.

($\leftarrow$) If $x = r(y, z)$ for some y, $z \in U$, then $x = g(\langle f^{-1}(y), f^{-1}(z)\rangle) \in K$. $\square$

Let $h : \mathcal{P}(U' \times U') \rightarrow \mathcal{P}(K)$ be the mapping induced by g, that is,

$$h(X) \stackrel{\text{def}}{=} \{g(x) : x \in X\} \text{ for all } X \subseteq U' \times U'.$$

Then h is one-one and onto, and

$$h(1^{\mathfrak{A}}) = h(U' \times U') = K = (1^*)^{\Omega^+}.$$

Define

$$\begin{aligned} O &\stackrel{\text{def}}{=} h(0^{\mathfrak{A}}) \\ P &\stackrel{\text{def}}{=} h(p^{\mathfrak{A}}) \\ Q &\stackrel{\text{def}}{=} h(q^{\mathfrak{A}}) \\ I &\stackrel{\text{def}}{=} h([1']^{\mathfrak{A}}) \\ V_i &\stackrel{\text{def}}{=} h(v_i^{\mathfrak{A}}) \qquad \text{if } 1 \leq i \leq k. \end{aligned}$$

Proposition 5.1.6.

$$\Omega^+ \models \varphi_3 \wedge \cdots \wedge \varphi_6 \wedge \forall x\big((\mathbf{V}_1(x) \vee \cdots \vee \mathbf{V}_k(x)) \rightarrow \mathbf{K}(x)\big).$$

Proof of Prop.5.1.6. In the course of the proof we will make use of the fact that f is onto. $h(0^{\mathfrak{A}}) = h(\emptyset) = \emptyset$, thus φ_3 is true.

$h(p^{\mathfrak{A}}) = h(\{\langle\langle u, v\rangle, u\rangle : u, v \in U'\}) = \{r(r(f(u), f(v)), f(u)) : u, v \in U'\} = \{r(r(y, z), y) : y, z \in U\}$, thus φ_4 is true, and a similar argument proves φ_5.

φ_6 is true since $h([1']^{\mathfrak{A}}) = h(\{\langle u, u\rangle : u \in U'\}) = \{r(f(u), f(u)) : u \in U'\} = \{r(y, y) : y \in U\}$.

Finally, $h(v_i^{\mathfrak{A}}) \subseteq K$ for all $1 \leq i \leq k$. $\Box$

In order to show that $(\tau_1^*)^{\Omega^+} \neq (\tau_2^*)^{\Omega^+}$ we need to estabilish the following

Proposition 5.1.7. If $\tau \in TRM^k_{\mathsf{TPA}}$, then

$$h(\tau^{\mathfrak{A}}) = (\tau^*)^{\Omega^+}.$$

Proof of Prop.5.1.7. We have already seen that $h(1^{\mathfrak{A}}) = (1^*)^{\Omega^+}$. For the other atomic terms τ in TRM^k_{TPA}, we have $h(\tau^{\mathfrak{A}}) = (\tau^*)^{\Omega^+}$ by definition (of Ω^+), so suppose that the statement holds for τ,$\mu \in TRM^k_{\mathsf{TPA}}$. Then

$$\begin{aligned} h([\sim\tau]^{\mathfrak{A}}) &= h(\{u \in U' \times U' : u \notin \tau^{\mathfrak{A}}\}) = \{g(u) : u \in U' \times U' \text{ and } u \notin \tau^{\mathfrak{A}}\} \\ &= \{x \in K : x \notin (\tau^*)^{\Omega^+}\} = (\lambda x(\mathbf{K}(x) \wedge \neg\tau^*(x)))^{\Omega^+} = ([\sim\tau]^*)^{\Omega^+}, \end{aligned}$$

$$\begin{aligned} h([\tau \cup \mu]^{\mathfrak{A}}) &= h(\{u \in U' \times U' : u \in \tau^{\mathfrak{A}} \cup \mu^{\mathfrak{A}}\}) \\ &= \{g(u) : g(u) \in (\tau^*)^{\Omega^+} \cup (\mu^*)^{\Omega^+}\} \\ &= \{x : x \in (\tau^*)^{\Omega^+} \cup (\mu^*)^{\Omega^+}\} = (\lambda x(\tau^*(x) \vee \mu^*(x)))^{\Omega^+} \\ &= ([\tau \cup \mu]^*)^{\Omega^+}, \end{aligned}$$

$$\begin{aligned} h([\tau^{-1}]^{\mathfrak{A}}) &= h(\{\langle v, u\rangle : \langle u, v\rangle \in \tau^{\mathfrak{A}}\}) \\ &= \{r(f(v), f(u)) : r(f(u), f(v)) \in (\tau^*)^{\Omega^+}\} \\ &= \{r(z, y) : r(y, z) \in (\tau^*)^{\Omega^+}\} \\ &= (\lambda x \exists y \exists z(x = r(y, z) \wedge \tau^*(r(y, z))))^{\Omega^+} \\ &= ([\tau^{-1}]^*)^{\Omega^+}, \end{aligned}$$

$$\begin{aligned} h([\tau \circ \mu]^{\mathfrak{A}}) &= h(\{\langle u, v\rangle : \exists s(\langle u, s\rangle \in \tau^{\mathfrak{A}} \text{ and } \langle s, v\rangle \in \mu^{\mathfrak{A}})\} \\ &= \{r(f(u), f(v)) : \exists s(r(f(u), f(s)) \in (\tau^*)^{\Omega^+} \text{ and} \\ &\qquad\qquad r(f(s), f(v)) \in (\mu^*)^{\Omega^+})\} \\ &= \{r(x, y) : \exists z(r(x, z) \in (\tau^*)^{\Omega^+} \text{ and } r(z, y) \in (\mu^*)^{\Omega^+})\} \\ &= (\lambda w \exists x \exists y \exists z(w = r(x, y) \wedge \tau^*(r(x, z)) \wedge \mu^*(r(z, y))))^{\Omega^+} \\ &= ([\tau \circ \mu]^*)^{\Omega^+}. \quad \Box \end{aligned}$$

We conclude that

$$(\tau_1^*)^{\Omega^+} = h(\tau_1^{\mathfrak{A}}) \neq h(\tau_2^{\mathfrak{A}}) = (\tau_2^*)^{\Omega^+},$$

so that

$$\Omega^+ \not\models \psi_e.$$

We have proved Theorem 5.1 $\Box$

6 $\mathrm{Eq}(\mathsf{TPA}^-)$ is recursively enumerable

Next we show that the RA-reducts of TPA's (called TPA^-) are much simpler, i.e., they admit a recursive axiomatization. This is so because the class TPA^- coincides with the class of those QRA's whose base is not a singleton. In more detail:

6.1 DEFINITION $\mathfrak{A} = \langle \mathcal{P}(U \times U), \ldots \rangle$ is a TPA^- iff $\langle \mathcal{P}(U \times U), \ldots p, q \rangle$ is a TPA. $\triangle$

6.2 LEMMA *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be full* RRA*'s (see [15, §5.3.2]) and suppose that their bases have the same cardinality. Then $\mathfrak{A} \cong \mathfrak{A}'$.*

Proof. The mapping induced by the function $f : U \longrightarrow U'$ is clearly an RA isomorphism. $\square$

6.3 LEMMA *U is empty or $|U| \geq \omega$ iff there is an U' with $|U| = |U'|$ and $U' \times U' \subseteq U'$.*

6.4 LEMMA *$\mathfrak{A}$ is a full* RRA *with empty or infinite base iff there is a $\mathfrak{B} \in \mathsf{TPA}^-$ such that $\mathfrak{A} \cong \mathfrak{B}$. Thus*

$$\mathbf{I}\{\textit{full } \mathsf{TPA}^-\} = \mathbf{I}\{\textit{full } \mathsf{RRA} \textit{ with empty or infinite base}\}.$$

Proof. By Lemmas 6.2 and 6.3. $\square$

Now consider the class of Q-Relation Algebras introduced in [40, 8.4]. Since $\mathsf{QRA} \subseteq \mathsf{RRA}$ by [40, 8.4 (iii)], it makes sense to speak of full QRA's, and by [40, 8.4 (iv)] we have

(11) {full QRA's with nonsingleton base} = {full RRA's with empty or infinite base}.

Combined with Lemma 6.4, this gives

(12) $\mathsf{TPA}^- \models e \quad \text{iff} \quad \{\text{full } \mathsf{QRA}\} \models [(0{=}1) \vee (1{\neq}\mathrm{Id})] \rightarrow e.$

6.5 LEMMA $\mathsf{QRA} = \mathbf{SP}\{\textit{full } \mathsf{QRA}\}$.

Proof. Let $\mathfrak{A} \in \mathsf{QRA}$. Then there are I and full RRA's $\mathfrak{B}_i$ $(i \in I)$ with

$$\mathfrak{A} \subseteq \mathfrak{A}^+ \cong \prod_{i \in I} \mathfrak{B}_i.$$

Now $\mathfrak{A}^+$ has conjugated quasiprojections in its universe (since $\mathfrak{A}$ has), and thus the $\mathfrak{B}_i$'s (being homomorphic images of $\mathfrak{A}$) are (full) QRA's. (Recall that the property of being a pair of conjugated quasiprojection is defined by an equation.) $\square$

Since $\mathrm{Eq}(\mathsf{QRA})$ is recursively enumerable, we have

6.6 THEOREM Eq(TPA^-) *is recursively enumerable.*

Proof. By (12) and Lemma 6.5, $\mathsf{TPA}^- \models e$ iff $\mathsf{QRA} \models (0{=}1) \vee (1{\neq}\mathrm{Id}) \to e$. $\square$

6.7 THEOREM TPA^- *is not finitely based.*

Proof. By the above Lemma 6.4 and (11) we have

$$\mathbf{I}\{\text{full } \mathsf{QRA}\} = \mathbf{I}(\{\text{full } \mathsf{TPA}^-\} \cup \{\mathfrak{B}\}),$$

where

$$\mathfrak{B} \stackrel{\text{def}}{=} \left\langle \{\emptyset, \{\langle a,a \rangle\}\}, \cup, \cap, \sim, 0, 1, \circ, {}^{-1}, 1' \right\rangle,$$

that is, $\mathfrak{B}$ is the QRA with singleton base. By Lemma 6.5 this means that

$$\mathsf{QRA} \models e \quad \text{iff} \quad (\mathsf{TPA}^- \models e \wedge \mathfrak{B} \models e).$$

Note that $\mathfrak{B}$ is term-equivalent to the two-element Boolean algebra, so $\mathfrak{B}$ is axiomatized by the finite set $\{\mathsf{BA}\text{-axioms}, 1 = 1', x \circ y = x \cap y, x^{-1} = x\}$.

Now suppose that $\{e_0, \ldots, e_{n-1}\}$ is a basis for TPA^-. Let $\theta \stackrel{\text{def}}{=} 0 \neq 1 \wedge 1 = 1'$ (note that θ holds in a full RRA iff its base is a singleton). Let

$$E \stackrel{\text{def}}{=} \{\neg\theta \to e_i : i < n\}$$

$$E' \stackrel{\text{def}}{=} \{\theta \to e'_i : i < m\},$$

where $\{e'_i : i < m\}$ is a basis for the equational theory of $\mathfrak{B}$. Then $E \cup E'$ is a finite equational basis for QRA, contradicting [40, 8.4(vii)]. Indeed, $\mathsf{QRA} \models E \cup E'$, since

$$\mathfrak{A} \in \text{full } \mathsf{QRA} \text{ and } \mathfrak{A} \models \neg\theta \text{ implies } \mathfrak{A} \in \mathsf{TPA}^-,$$

whence $\mathfrak{A} \models e_i$ if $i < n$, and

$$\mathfrak{A} \in \text{full } \mathsf{QRA} \text{ and } \mathfrak{A} \models \theta \text{ implies } \mathfrak{A} \cong \mathfrak{B},$$

thus $\mathfrak{A} \models e'_i$ if $i < m$. On the other hand, suppose that $\mathsf{QRA} \models e$. Then $\mathsf{TPA}^- \models e$ and $\mathfrak{B} \models e$, so $\{e_i : i < n\} \vdash e$ and $\{e'_i : i < m\} \vdash e$, and thus $E \vdash \neg\theta \to e$ and $E' \vdash \theta \to e$, giving $E \cup E' \vdash e$. $\square$

7 Related work

For the interested reader, here are some more remarks concerning work related to our paper.

1) It turned out that complexity issues, and representability of certain classes of algebras depend on the choice of the underlying set theory. *Non-well-founded set theories* were investigated, from this point of view, in e.g. Kurucz-Németi [19], Sain-Németi [36]. One of the referees of this paper pointed out that it would be interesting to investigate whether our Thm.4.1 would change if we replaced ZFC with some non-well-founded set theory (since it is clear that the axiom of foundation has been used in our proof).

2) Both of our referees pointed out that *directed cylindric algebras* $\mathsf{CA}^{\uparrow}$ have been introduced for providing a cylindric algebraic analogue of quasi-projective relation algebras, see e.g. Németi-Simon [28] and Sági [30]. Therefore it would be a nice future project to find a cylindric algebraic version of what is going on in this paper, implemented via $\mathsf{CA}^{\uparrow}$.

3) Connecting to this, Gábor Sági called our attention to the following. According to our Theorems 6.6 amd 6.7, the equational theory of TPA^{-} is recursively enumerable, but not finitely based. This result is in interesting contrast with the following one. For finite $n > 3$, the equational theory of $\mathsf{CA}^{\uparrow}$ is finitely based, and in some non-well-founded set theories, each $\mathsf{CA}^{\uparrow}$ is representable as a set algebra (such that each operation is a concrete, set theoretical one, and they are invariant for the automorphisms of the set theoretical universe).

4) As we already mentioned in subsection 2.3 item 2), in [40], relation algebras have been used for formalizing set theory. Some improvements of the results of [40] can be found in Kurucz [18] and in the more recent paper [2] by Andréka and Németi. In both papers, weaker theories of relation algebras were established – to strengthen the original results.

5) In his dissertation [11] and in related papers (e.g. [10]), Miklós Ferenczi presented a representation theory of cylindric-like algebras, based on relativized set algebras (instead of the “square” ones used traditionally). As a side effect of this, he proved a family of new results solving the finitization problem in new cases (truly finite axiomatizations for finite dimensional cases).

6) One of our referees called our attention to the fact that Tarek Sayed-Ahmed [38] approaches the finitization problem of first order logic FOL *with equality* in a new way. Namely, [38] combines the ideas of Miklós Ferenczi on *representations by relativized set algebras* (cf. the previous item 5)), and the *semigroup approach* adopted by Ildikó Sain in her solution to the finitization problem for FOL *without equality* (cf. e.g. [35]). This way (similarly to the solution in [35]), [38] gives *truly finite axiomatizations even in infinite dimensional cases.*

7) We mention but do not discuss here that Balázs Biró [6] gave a noteworthy negative result concerning one particular way of trying to solve the finitization problem (permutation invariance assumed).

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